Commodity Money, Free Banking, and Nominal Income Targeting

Lessons for Monetary Policy Reform

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Abstract

Some have argued that nominal income targeting is desirable because it would replicate characteristics of a free banking regime. However, the degree to which this is true and desirable depends on the properties of commodity-based monetary regimes. In this paper, I provide a model of commodity money. I find that a pure commodity money regime can only generate an efficient stationary equilibrium by divine coincidence or by giving policymakers control over the supply of the commodity. The introduction of bank notes makes it much more likely that the economy will achieve an efficient equilibrium. In particular, in a commodity-based system, bank notes are equivalent to call options on the commodity, and the commodity holdings are equivalent to a put option on the commodity. Assuming that there is no risk-free arbitrage in equilibrium, then both bank notes and the commodity will have an expected rate of return equal to the risk-free rate. If the risk-free rate is equal to the rate of time preference, then this commodity regime is efficient. A free banking system would therefore not only minimize deviations between the supply and demand for money, but it would also (potentially) implement the Friedman rule. Both market-based and more conventional nominal income targeting regimes are unlikely to replicate both features of a free banking system unless the nominal income target has a deflationary bias.

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Introduction

Commodity money regimes come in a variety of forms. One possible regime is one in which the commodity itself circulates (e.g., gold, silver, or even copper coins). Another regime is one in which bank notes circulate, but they are redeemable in terms of the commodity. It is also possible to have a mixture of these two regimes, in which both precious metal coins and bank notes circulate alongside one another. Even within a particular regime, there might be institutional differences. For example, bank notes could be issued by individual banks in a competitive market or by a central bank with a monopoly over note issuance.

While these commodity money regimes have disappeared, a number of proposals for monetary policy reform draw on lessons or characteristics from commodity money regimes. For example, Thompson (1974), Glasner (1989), Selgin (1988, 1994), and Hendrickson (2015) have each argued that a free banking regime would eliminate the adverse consequences of deviations between the money supply and money demand.[1] Some advocates of nominal income targeting have argued that it would replicate this feature of free banking by automatically adjusting the money supply in accordance with money demand (Selgin 1994; Horwitz 2000; Hendrickson 2012, 2015).

[1] A note about language: The term *free banking* refers to a banking system and not a monetary system. Nonetheless, to economize on words, *free banking* in this paper refers to a system in which banks competitively supply bank notes that are redeemable in terms of a real commodity.

This literature emphasizes the stabilization properties of free banking and the ability of a nominal income target to replicate these properties, but it does not say much about the precise target for nominal income and how it is determined. In this paper, I argue that a characteristic that has heretofore been overlooked is that a commodity money regime in which bank notes are convertible into the commodity has the potential to automatically achieve long-run efficiency. Not only would a free banking regime result in the money supply adjusting in response to changes in money demand, but it could also eliminate long-run inefficiencies because it offers an expected return on bank notes equal to the risk-free rate. The competitive money supply issuance ensures that the money supply adjusts in accordance with money demand, thereby eliminating monetary disequilibria—deviations from the steady state. By issuing bank notes with an option value, commodity money regimes produce a positive (implicit) expected rate of return on money. This (potentially) achieves the efficient steady state equilibrium. Advocates of nominal income targeting have focused on the first characteristic with little attention given to the second.

The purpose of this paper is to develop a model of commodity money, examine the long-run efficiency properties of a pure commodity money regime, and contrast this regime with a regime that has bank notes that are convertible into the commodity at a fixed price. This comparison helps to elucidate the source of long-run efficiency in convertible paper money regimes. The case for a monetary policy regime that replicates the features of a free banking system is stronger than scholars have previously argued. The results also have direct implications for determining a precise target for nominal income when implementing a nominal income targeting regime, whether it is implemented within the current institutional regime or through a market-based approach. A characteristic of each regime is that the nominal income target should have a deflationary bias.

To examine the efficiency properties of commodity money regimes to draw lessons for contemporary proposals for monetary reform, I develop a model of commodity money within a continuous time version of the monetary search model of Lagos and Wright (2005). The model is closest to that used by Hendrickson and Salter (2016). It includes three possible media of exchange: commodity money, claims to a risky asset, and risk-free government bonds. Commodity money is assumed to be perfectly recognizable, whereas claims to the risky asset and bonds are not. However, everyone in the model economy can pay a transaction cost proportional to a buyer's offer of the relevant asset to verify the asset's authenticity. I use the model to derive equilibrium conditions and solve for the rates of return on each asset, the volatility of the purchasing power of the commodity, and the volatility of claims to the risky asset. I then use the model to discuss the efficiency of a pure commodity money equilibrium (an equilibrium in which the commodity itself—and not bank notes—circulates as a medium of exchange). A pure commodity money regime can only generate the efficient equilibrium allocation by (1) divine coincidence, or (2) giving policymakers direct control over the supply of the commodity.

The introduction of bank notes that are redeemable for a fixed quantity of the commodity represents a possible welfare improvement over a pure commodity money regime. In a system in which bank notes are redeemable for a fixed quantity of the commodity, these bank notes are equivalent to banks selling call options for the commodity. Using standard option pricing assumptions, the expected rate of return on the option is equal to the risk-free real interest rate. It therefore follows that bank notes that are redeemable in terms of the commodity pay an implicit expected rate of return equal to the risk-free rate. In other words, in a commodity money regime, bank notes earn a positive expected rate of return. This result is in contrast to an inconvertible paper money regime, in which paper money earns a real rate of return equal to the negative inflation rate. The crucial difference is that under a commodity money regime, the bank promises to redeem bank notes for a given quantity of the commodity, whereas under an inconvertible paper money regime, the bank promises to redeem bank notes or deposit values for an asset of equal *nominal* value.

Free Banking, Nominal Income Targeting, Stabilization, and Efficiency

The purpose underlying proposals to target nominal income is to prevent monetary disequilibrium. The idea is that the demand for money is a demand for real money balances. If the nominal supply of money at the current price level differs from the real quantity of money that the public wants to hold, this is referred to as monetary disequilibrium. When the real value of the nominal supply of money in circulation is less than the desired real money balances of the public, there is an excess demand for money. Since money is traded in all markets, an excess demand for money corresponds to an excess supply of goods. As a result, spending declines, reducing output and prices. This process continues until prices fall sufficiently to equate the real value of the money supply with desired real money balances. To the extent to which this mechanism is to blame for business cycle fluctuations, a monetary system that could eliminate such disequilibria would significantly improve welfare.[2]

Under a free banking regime, banks promise to redeem bank notes for a fixed quantity of a particular commodity. The nominal price of the commodity is therefore fixed. At the same

[2] While the concept of monetary disequilibrium is often associated with Old Monetarists like Warburton (1950), Friedman (1956), and Yeager (1997), it can be actually be found throughout a wide economic literature and even stretches across divergent schools of thought, such as the Austrian business cycle theory and the branch of Keynesianism associated with the study of coordination problems (Horwitz 2000). For further exposition of Friedman's views and an empirical analysis thereof, see Hendrickson (2017).

time, the supply and demand for the commodity determines the relative price of the commodity (the price of the commodity in terms of all other goods). Given that the nominal price of the commodity is fixed, it follows that the price level is determined by the supply and demand for the commodity (Barro 1979). Banks within the free banking system are then willing to supply any quantity of bank notes that the public wants to hold at that price level. In other words, the supply of bank notes is perfectly elastic. To understand why the supply curve for bank notes is perfectly elastic, consider that banks are motivated by profit considerations to expand the size of their balance sheet and therefore increase their bank note issuance. However, any increase in the supply of bank notes above the amount the public is willing to hold will return to the overissuing banks through the law of reflux (Glasner 1989, 1992).[3] This outcome implies that "shocks" to the supply of bank notes will not have any effect on the price level or real economic activity. Furthermore, since the supply curve for bank notes is perfectly elastic, the supply of bank notes adjusts in accordance with changes in the demand for bank notes. As a result, "shocks" to money demand are similarly irrelevant for macroeconomic fluctuations. The elimination of deviations between the demand for and the supply of money would seem to be somewhat significant, given that such deviations are at the center of a wide variety of monetary theories of the business cycle.

The idea that a nominal income targeting regime would replicate this outcome can be understood by appealing to the equation of exchange, $MV = PY$, where M is the money supply, V is the velocity of circulation, and PY is nominal income. Within this context, any fluctuation in V represents an inverse change in the demand for money. To ensure that the money supply

[3] The term "law of reflux" has often been associated with the real bills doctrine. Others have eschewed the use of this term because of its association with the real bills doctrine. I do not begrudge them. Nonetheless, I am using the term "law of reflux" as it is defined by Glasner (1992, 869): "What the law of reflux asserts is that private banks cannot create an inflationary overissue, because there is a market mechanism that induces banks to supply just the amount of money that the public wants to hold."

moves in conjunction with money demand, a central bank would want to adjust M to keep the product, MV, constant. Since it would be difficult in practice for the central bank to predict changes in V and adjust M accordingly, an alternative way of implementing this policy would be to target nominal income.

This argument emphasizes the desirability of a nominal income target for reducing economic fluctuations at business cycle frequencies. In other words, this characteristic of nominal income targeting is one that minimizes deviations from the steady state. Notably absent from these arguments is any reference to the desirability or efficiency of the long-run steady state itself. This distinction is important because the existing arguments make the case for nominal income targeting on the basis of its stabilization properties, but they do not address the issue of what the precise target should be.

In formulating a target for nominal income growth, one might construct the target as the sum of growth in the trend of real GDP and the optimal rate of inflation. There is a large literature on the optimal rate of inflation.[4] Within this literature, a common result is that the Friedman rule is optimal. According to the Friedman rule, the central bank should pay interest on cash balances at a rate equal to the market interest rate or enact a policy of deflation in order to drive the nominal market interest rate to zero. This is optimal because in a world with fiat money, the nominal interest rate on cash is zero. Other assets pay interest but are only imperfect substitutes for cash; this creates an inefficiency. The cost of holding cash causes individuals to economize on cash balances in favor of interest-bearing assets that are less liquid. This outcome reduces the feasible opportunity set for trade and precludes Pareto efficiency.

[4] For a useful overview of recent literature, see Diercks (2017).

Calls for nominal income targeting come in a variety of forms. For example, some would prefer to implement a nominal income target within the confines of the current institutional environment. Others would prefer to implement a nominal income target or a nominal wage target by targeting futures contracts or through indirect convertibility (Sumner 1989, 2013; Thompson 1982; Glasner 1989).[5] Within the current institutional environment, it might be sufficient to simply rely on the Friedman rule for formulating an optimal rate of inflation to determine the precise nominal income target. However, the same logic does not necessarily apply to the market-based proposals. By targeting futures contracts or by using indirect convertibility, these proposals likely have much more in common with free banking under a commodity standard than they do with the current institutional regime. It is therefore important to consider the efficiency properties of a free banking regime in determining the precise nominal income target.

To my knowledge, little attention has been paid to the efficiency properties of steady-state equilibria in models with commodity money in a general equilibrium setting. Nonetheless, one might argue that Hotelling's rule implies that the rate of return on an exhaustible resource will equal the real interest rate (Rockoff 1984). In a Hotelling model, given the price path over time, the resource owner can choose the amount of time to wait to extract the resource. This maximization problem yields a solution in which the rate of return on the exhaustible resource is equal to the real rate of interest. The basic intuition is that when the value of the exhaustible resource is growing at a rate below the market rate of interest, it makes sense to extract the resource, sell it, and purchase the interest-bearing asset. If, on the other hand, the value of the asset is growing faster than the rate of interest, it is better not to extract the resource and let its value grow.

[5] I discuss these proposals at greater length below.

The problem with applying a Hotelling-type model is that the Hotelling rule really implies that the rate of return on the option to extract the resource is equal to the real rate of interest. This type of model does not pin down the rate of return on the resource itself. In fact, the rate of return on the underlying resource is often taken as given.[6]

As with any resource, the rate of return on the exhaustible resource is determined by the supply and demand for the resource. When an exhaustible resource is valued as a medium of exchange, at any given point in time there will be a stock demand and a stock supply of the asset. The stock demand for the asset is determined by the portfolio decisions of individuals. At the same time, the stock supply of the asset is affected by the production flow over time. The relative price of the commodity is determined by the stock-flow equilibrium. The real demand for the stock of the commodity is determined by those using it as a medium of exchange. This demand will be constant in a stationary equilibrium. Yet the actual stock might be increasing or decreasing, depending on the net changes in the flow supply and demand for the commodity owing to things like gold discoveries, changes in input costs, or changes in consumption demand for the commodity. Thus, the rate of return on the commodity (in terms of goods) is determined by changes in the net flow of the commodity over time.

Finally, there is no guarantee that this rate of return will be equal to the real rate of interest. The only way to guarantee that the rate of return on the commodity will equal the real rate of interest is to have complete, arbitrage-free markets. A standard result from neoclassical finance is that this can only occur if there is a contingent claims contract in which the number of underlying assets is equal to the number of "shocks," or sources of uncertainty. As I demonstrate below, this condition holds when banks issue notes that are redeemable in terms of the

[6] I discuss this in more detail in the appendix.

commodity at a fixed price. This occurrence creates a particular contingent claim known as a perpetual American call option on the commodity. Since the rate of return on the commodity is subject only to shocks to the net flow of the commodity into use as a medium of exchange, and since there is one underlying asset (the commodity) for this contingent claim, the market is complete and arbitrage free. The commodity and the bank note therefore have an expected rate of return equal to the real interest rate.

This discussion emphasizes the key results of the model that follows. In the absence of bank notes, there is no guarantee that a commodity money regime achieves the efficient stationary equilibrium allocation. The introduction of redeemable bank notes resolves this problem by creating a complete market. Thus, the benefit of a free bank system is not only the stabilization properties of the system, but also the fact that such a system can potentially achieve the efficient allocation automatically. This latter result is owing to the convertibility of notes. The case for a monetary regime that replicates the characteristics of a free banking regime is stronger than scholars have previously argued. When implementing market-based nominal income targeting regimes, this consideration needs to be taken into account. In what follows, I demonstrate this formally with a model and then discuss the implications for monetary policy reform.

Model

Time is continuous and infinite. There is a continuum of individuals with unit mass. Ordinarily, the individuals are located in a centralized market where they can produce and consume. However, while in the centralized market, they randomly bump into other individuals for potential pairwise meetings to trade. When individuals are matched pairwise, they negotiate the terms of trade. Different goods are traded in the centralized market and pairwise meetings. These goods are hereafter referred to as the centralized market good and the search good. In the centralized market, all individuals can produce the centralized market good. In pairwise meetings, individuals lack the production technology to make the centralized market good and cannot access the centralized market. In these pairwise meetings, individuals can produce the search good.

In pairwise meetings, one individual is a buyer and one individual is a seller; each individual lacks information about the trading history of the counterparty. As a result, a medium of exchange is essential. There are three possible media of exchange in the model. The first is a commodity money that is perfectly recognizable and costless to exchange. The two other possible media of exchange are government bonds and claims to a real asset that has a fixed supply. What makes the government bonds and claims to the real asset different from the commodity money is that neither the bonds nor the claims are perfectly recognizable. As a result, it is possible to counterfeit bonds and claims to the real asset. To avoid counterfeiting, sellers can pay a cost that is proportional to the asset claims or bonds that are being offered. If sellers pay the cost, they never accept a counterfeit. As such, while there is a threat of counterfeiting, there is no actual counterfeiting in equilibrium.

The preferences of the individuals in the model economy are given as

$$\mathcal{U} = \int_0^\infty e^{-\rho t}\{u[q^b(t)] - c[q^s(t)] + x(t) - y(t)\}dt \tag{1}$$

where q^b is the quantity of the search good consumed by individuals as buyers, q^s is the quantity of the search good produced by individuals as sellers, x is the quantity of the centralized market good consumed, $y \leq \bar{y}$ is the quantity of the centralized market good produced by the individual (where $\bar{y}$ is defined to be sufficiently large that this constraint never binds), ρ is the rate of time preference, $u', -u'' > 0$, $c' > 0$, and $c'' \geq 0$.[7]

The supply of the commodity is assumed to follow a geometric Brownian motion:

$$\frac{dG^A}{G^A} = \mu dt + \sigma dz \tag{2}$$

where G^A is the supply of the commodity, μ is the expected rate of change in the supply, σ is the standard deviation, and dz is an increment of a Wiener process. Both μ and σ are exogenous.

Finally, there is also a government that makes lump-sum transfers (or collects lump-sum taxes). The government budget constraint is

$$db^A = (rb^A + \gamma)dt \tag{3}$$

where b^A is the real quantity of aggregate debt, r is the risk-free interest rate paid on the debt, and γ is the real value of the government transfer (or tax, if negative).

Centralized Market

Let each individual be indexed by i, G_i denote the quantity of the commodity measured in ounces, and ϕ denote the price of the commodity in terms of the centralized market good. It follows that $g_i = \phi G_i$ is the real value of the commodity in terms of goods. Let's conjecture that the price of the commodity in terms of goods follows a geometric Brownian motion:

$$\frac{d\phi}{\phi} = \alpha_\phi dt + \sigma_\phi dz \tag{4}$$

[7] The assumption that production is unconstrained is necessary to ensure that the distribution of money balances is degenerate in continuous time. For a discussion of such issues, see Rocheteau, Weill, and Wong (2015).

where α_ϕ and σ_ϕ are parameters. This conjecture implies that the parameter α_ϕ measures the expected rate of change in the price of the commodity in terms of the centralized market good. Recall that dz is the shock to the supply of the commodity. The parameter σ_ϕ therefore measures the responsiveness of the price of the commodity in terms of the centralized market good to this aggregate shock. These parameters represent a conjecture and are therefore endogenously determined by the model.

The nominal value of the commodity owned by an individual has the following law of motion:

$$dG_i = (Y_i + \Gamma_i + rB_i - X_i - N_{i,b} - N_{i,a})dt$$

where Y is the value of production in terms of the commodity, Γ is the value of the transfer in terms of the commodity, B_i is the value of bonds in terms of the commodity, X is the value of consumption in terms of the commodity, N_b is the value of net bond purchases in terms of the commodity made by the individual, and N_a is the value of net real asset purchases in terms of the commodity. Multiplying both sides by ϕ and using lowercase letters to denote real values (the value in terms of the centralized market good), this relationship can be written as

$$\phi dG_i = (y_i + \gamma_i + rb_i - x_i - n_{i,b} - n_{i,a})dt \tag{5}$$

Note that $dg_i = d(\phi G_i) = G_i d\phi + \phi dG_i$. Using equations (4) and (5), it follows that

$$dg_i = (y_i + \gamma_i + rb_i - x_i - n_{i,b} - n_{i,a} + \alpha_\phi g_i)dt + \sigma_\phi g_i dz \tag{6}$$

Similarly, let's conjecture that the price of the real asset in terms of the centralized market good also follows a geometric Brownian motion, such that

$$\frac{dP_a}{P_a} = \alpha_a dt + \sigma_a dz \tag{7}$$

where P_a is the price of the asset in terms of the centralized market good and α_a and σ_a are parameters. Here, α_a is the expected rate of change in the price of the asset in terms of the centralized market good, and σ_a is the responsiveness of the asset price to the shock to the supply of the commodity. Again, as in the case of commodity money, these parameters are endogenously determined by the model.

Using the same logic as above, it follows that the evolution of the real asset can be written as

$$da_i = (n_{i,a} + \alpha_a a_i)dt + \sigma_a a_i dz \tag{8}$$

where a is the real value of the asset.

Finally, since interest payments on bonds are paid in the form of the commodity, the evolution of real bond balances is given as

$$db_i = n_{i,b}dt \tag{9}$$

Let $F(t)$ denote the cumulative distribution function that measures the probability that an individual has been matched pairwise to trade at or before period t, and let $f(t)$ denote the corresponding density function. Also, let $W(g_i, a_i, b_i)$ denote the value function of individual i in the centralized market. It follows that

$$W(g_i, a_i, b_i) = \max_{y_i, x_i, n_{i,b}, n_{i,a}} \int_0^\infty e^{-\rho t}\{[1 - F(t)][x_i(t) - y_i(t)] + f(t)V(g_i, a_i, b_i)\}dt$$

where $V(\cdot)$ denotes the value function associated with pairwise meetings.

I assume that the amount of time it takes to be matched pairwise is drawn from an exponential distribution with an arrival rate θ. It follows that

$$F(t) = 1 - e^{-\theta t}$$

$$f(t) = \theta e^{-\theta t}$$

Plugging this into the value function yields

$$W(g_i, a_i, b_i) = \max_{y_i, x_i, n_{i,b}, n_{i,a}} \int_0^{\infty} e^{-(\rho+\theta)t}\{[x_i(t) - y_i(t)] + \theta V(g_i, a_i, b_i)\}dt$$

Bellman's equation can be written as

$$(\rho+\theta)W(g_i, a_i, b_i) = \max_{y_i, x_i, n_{i,b}, n_{i,a}} x_i - y_i + \theta V(g_i, a_i, b_i) + \frac{1}{dt}EdW \tag{10}$$

Using Ito's Lemma,

$$dW = W_g dg_i + W_a da_i + W_b db_i + \frac{1}{2}W_{gg}(dg_i)^2 + \frac{1}{2}W_{aa}(da_i)^2 + \frac{1}{2}W_{bb}(db_i)^2$$

Using equations (6), (8), and (9), combined with Ito's Lemma, I can rewrite equation (10) as

$$(\rho+\theta)W(g_i, a_i, b_i) = \max_{y_i, x_i, n_{i,b}, n_{i,a}} x_i - y_i + \theta V(g_i, a_i, b_i) + W_g(y_i + \gamma_i + rb_i - x_i - n_{i,b} - n_{i,a} + \alpha_\phi g_i) +$$

$$W_a(n_{i,a} + \alpha_a a_i) + W_b n_{i,b} + \frac{1}{2}W_{gg}\sigma^2 g_i^2 + \frac{1}{2}W_{aa}\sigma^2 a_i^2 \tag{11}$$

The first-order conditions for this maximization problem are given as

$$W_g = 1 \tag{12}$$

$$W_g = W_a \tag{13}$$

$$W_g = W_b \tag{14}$$

The intuition of these first-order conditions is as follows. The first condition implies that the marginal value of holding an additional unit of the commodity is equal to the marginal utility of consuming the centralized market good. This is a standard condition of any life-cycle model. If the marginal value of the asset is greater than the marginal utility of consumption, then the individual should accumulate more of the asset. If the marginal value is less than the marginal utility, then the individual should use the asset to purchase some of the consumption good. The individual will continue this process until the marginal value of the commodity money is equal to the marginal utility of what the commodity money will purchase. The remaining conditions state that the marginal values of each of the assets are equal. Again, if the marginal values differed, then the individual would have an incentive to reallocate his or her portfolio in favor of the asset with the highest marginal value. This process would continue until the marginal values equalized.

Using the fact that $W_g = W_a = W_b = 1$, it follows that $W_{gg} = W_{aa} = 0$. Using these values, I can then compute the marginal value of each asset from equation (11). This calculation yields three equilibrium conditions:

$$\rho + \theta = \theta V_g + \alpha_\phi \tag{15}$$

$$\rho + \theta = \theta V_a + \alpha_a \tag{16}$$

$$\rho + \theta = \theta V_b + r \tag{17}$$

To generate a solution, we need to solve for the marginal value of each asset in pairwise meetings.

Pairwise Meetings

As alluded to above, there is some probability that individuals bump into possible trading partners in the centralized market. During the meetings, individuals can trade a different good, the search good. When these individuals are matched pairwise, they receive preference shocks and matching shocks. For simplicity, I lump these two shocks together and assume that the individual wants to be a buyer and is matched with a seller with probability 1/2 and that the individual wants to produce and is matched with a buyer with probability 1/2. It follows that the value function for those entering pairwise meetings can be written as

$$V(g_i, a_i, b_i) = \frac{1}{2}[u(q^b) + W(g_i - d_g, a_i - d_a, b_i - d_b)] + \frac{1}{2}[-c(q^s) + W(g_i + d_g, a_i + d_a, b_i + d_b)] \quad (18)$$

where $d_g \leq g_i$, $d_a \leq a_i$, and $d_b \leq b_i$ are the real quantities of the assets offered by the buyer in exchange for the search good. From the marginal conditions, above, we know that $W_g = W_a = W_b = 1$. Thus, W is linear. Given the linearity of W, the value function is given as

$$V(g_i, a_i, b_i) = \frac{1}{2}[u(q^b) - c(q^s)] + W(g_i, a_i, b_i) \quad (19)$$

From equations (15)–(17), I need to know the marginal value of each asset in pairwise meetings. From equation (19),

$$V_g = \frac{1}{2}[u'(q) - c'(q)]\frac{\partial q}{\partial g_i} + 1 \quad (20)$$

$$V_a = \frac{1}{2}[u'(q) - c'(q)]\frac{\partial q}{\partial a_i} + 1 \quad (21)$$

$$V_b = \frac{1}{2}[u'(q) - c'(q)]\frac{\partial q}{\partial b_i} + 1 \quad (22)$$

where I have used the fact that $q^b = q^s = q$ must be true in pairwise trade. To obtain the partial derivatives in the previous equations, I turn my attention to the bargaining problem in these pairwise meetings.

When individuals are matched, they negotiate the terms of trade. For simplicity, I assume that buyers make take-it-or-leave-it offers to sellers. In order for sellers to participate in the transaction, the offer made by buyers must be sufficient to induce sellers to produce. The incentive compatibility constraint is given as

$$d_g + (1 - \chi_a)d_a + (1 - \chi_b)d_b \geq c(q^s) \tag{23}$$

where $\chi_a \in (0, 1)$ and $\chi_b \in (0, 1)$ represent the transaction costs that the seller has to pay to verify the authenticity of claims to the real asset and bonds, respectively.

The buyer's problem is to choose q, d_g, d_a, and d_b to maximize

$$u(q^b) - d_g - d_a - d_b$$

subject to equation (23). The buyer's maximization problem is therefore written as

$$\max_{d_g, d_a, d_b} \mathcal{L} = u(q) - d_g - d_a - d_b + \lambda_1[d_g + (1-\chi_a)d_a + (1-\chi_b)d_b - c(q)] + \lambda_2(g_i - d_g) + \lambda_3(a_i - d_a) + \lambda_4(b_i - d_b)$$

where λ_1, λ_2, λ_3, and λ_4 are Lagrangian multipliers, and where I have dropped the superscript on q because $q^b = q^s$ in any two-party trade. The first-order conditions for maximization are

$$u'(q) = \lambda_1 c'(q) \tag{24}$$

$$\lambda_1 = 1 + \lambda_2 \tag{25}$$

$$(1 - \chi_a)\lambda_1 = 1 + \lambda_3 \tag{26}$$

$$(1 - \chi_b)\lambda_1 = 1 + \lambda_4 \tag{27}$$

Note that for any trade to take place, there must be some surplus to be divided. This implies that $u(q) \geq c(q)$. Differentiating both sides implies that $u'(q) \geq c'(q)$. From the first maximization condition, this implies that $\lambda_1 \geq 1$. This implies that the incentive compatibility constraint, equation (23), is always binding. Whether the remaining constraints are binding is an open question. For now, I will restrict attention to a scenario in which $\lambda_j > 0, \forall j$.

Proposition 1. Assume that $\alpha_\phi < \rho + \frac{\theta}{2} - \frac{\theta}{2(1-\chi_a)}$ and that the transaction cost for verifying the risky asset is higher than for the riskless bond ($\chi_a > \chi_b$). It follows that $d_g = g_i$, $d_a = a_i$, and $d_b = b_i$.

Proof. Recall from equation (15) that

$$\rho + \theta - \alpha_\phi = \theta V_g$$

Using equation (20) and the fact that $u'(q) = \lambda_1 c'(q)$, this can be written as

$$\rho - \alpha_\phi = \theta \frac{1}{2}[(\lambda_1 - 1)c'(q)]\frac{\partial q}{\partial g_i} \tag{28}$$

Note that, by assumption, $\alpha_\phi < \rho + \frac{\theta}{2} - \frac{\theta}{2(1-\chi_a)}$, and therefore it is necessarily true that $\alpha_\phi < \rho$. This implies that $\lambda_1 > 1$, and therefore $\lambda_2 > 0$, and therefore $d_g = g_i$. From the incentive feasibility constraint, it follows that

$$g_i + (1 - \chi_a)d_a + (1 - \chi_b)d_b = c(q)$$

Using the implicit function theorem,

$$\frac{\partial q}{\partial g_i} = \frac{1}{c'(q)}$$

Plugging this into equation (28) yields

$$\rho - \alpha_\phi = \frac{\theta}{2}(\lambda_1 - 1)$$

Solving for λ_1 yields

$$\lambda_1 = \frac{2(\rho - \alpha_\phi)}{\theta} + 1$$

Consider the equilibrium conditions for the offer of claims to the risky asset

$$\lambda_1 = \frac{1 + \lambda_3}{1 - \chi_a}$$

It follows that $\lambda_3 > 0$ if

$$\lambda_1 > \frac{1}{1 - \chi_a}$$

Or,

$$\frac{2(\rho - \alpha_\phi)}{\theta} + 1 > \frac{1}{1 - \chi_a}$$

This condition will be satisfied if

$$\rho + \frac{\theta}{2} - \frac{\theta}{2(1 - \chi_a)} > \alpha_\phi$$

This was my first assumption above. Thus, $\lambda_3 > 0$.

Now, combining the equilibrium conditions for the risky asset and the riskless bond, it follows that

$$\frac{1 - \chi_a}{1 - \chi_b} = \frac{1 + \lambda_3}{1 + \lambda_4}$$

Since my second assumption was that $\chi_a > \chi_b$, it follows that

$$\frac{1+\lambda_3}{1+\lambda_4} < 1$$

and therefore $\lambda_4 > \lambda_3 > 0$.

This proposition establishes that individuals will offer their entire asset holdings in equilibrium. Later, I will explore whether the assumption regarding α_ϕ is realistic. However, for now, I note that Proposition 1 implies that the incentive feasibility constraint, equation (23), can be written as

$$g_i + (1-\chi_a)a_i + (1-\chi_b)b_i = c(q) \tag{29}$$

Using the implicit function theorem, it follows that the marginal value conditions, equations (20)–(22), can be written as

$$V_g = \frac{1}{2}\left(\frac{u'(q)}{c'(q)} - 1\right) + 1$$

$$V_a = \frac{1-\chi_a}{2}\left(\frac{u'(q)}{c'(q)} - 1\right) + 1$$

$$V_b = \frac{1-\chi_b}{2}\left(\frac{u'(q)}{c'(q)} - 1\right) + 1$$

Plugging these marginal values into equilibrium conditions (15)–(17), respectively, yields

$$\rho - \alpha_\phi = \theta\frac{1}{2}\left(\frac{u'(q)}{c'(q)} - 1\right) \tag{30}$$

$$\rho - \alpha_a = \theta\frac{1-\chi_a}{2}\left(\frac{u'(q)}{c'(q)} - 1\right) \tag{31}$$

$$\rho - r = \theta\frac{1-\chi_b}{2}\left(\frac{u'(q)}{c'(q)} - 1\right) \tag{32}$$

No-Arbitrage Conditions

The equilibrium conditions above will hold if and only if individuals have an incentive to engage in transactions. To ensure that this is the case, I have to rule out the possibility of a perpetual money pump. In other words, if individuals in the model can construct a portfolio that earns positive profits without any risk, then they would want to pursue this strategy perpetually and allow their wealth to get arbitrarily large. In this section, I construct a hypothetical portfolio and derive a condition that must hold in order to rule out risk-free arbitrage. Since this is a model of the long run, I assume that arbitrage opportunities have been exhausted and impose this no-risk-free-arbitrage condition as an equilibrium condition in the model.

Suppose that an individual constructs a portfolio with n_g units of the commodity and n_a units of asset claims, and these units are fixed over time. It follows that the value of the portfolio in terms of the centralized market good can be written as

$$\mathcal{V} = \phi n_g + P_a n_a$$

The change in the value of the portfolio over time is

$$d\mathcal{V} = n_g d\phi + n_a dP_a$$

Or, defining $s := n_g/V$ as the share of the portfolio allocated to commodity money,

$$\frac{d\mathcal{V}}{\mathcal{V}} = [s\alpha_\phi + (1-s)\alpha_a]dt + [s\sigma_\phi + (1-s)\sigma_a]dz$$

Suppose that an individual chooses s to eliminate the uncertainty associated with the value of the portfolio. It follows that

$$s\sigma_\phi + (1-s)\sigma_a = 0$$

Solving for s yields

$$s = -\frac{\sigma_a}{\sigma_\phi - \sigma_a}$$

In order to rule out the possibility of a perpetual money pump, if the portfolio is riskless, it must earn the same rate of return as the riskless bond. Thus,

$$[s\alpha_\phi + (1-s)\alpha_a] = r$$

Using the solution for s, this can be rewritten as

$$\frac{\alpha_\phi - r}{\sigma_\phi} = \frac{\alpha_a - r}{\sigma_a} \tag{33}$$

This result implies that the Sharpe ratios for commodity money and for the claims to the real asset must be identical. In other words, the absence of arbitrage implies that individuals can only earn higher relative returns if they are willing to bear additional risk.

Equilibrium Conditions

In a stationary equilibrium, the bond supply must be constant. It follows that

$$rb^A = -\gamma$$

In equilibrium, the government levies a lump-sum tax to finance real interest payments.

In the centralized market, the market clearing condition is $\int_0^1 x_i di = x = \int_0^1 y_i di = y$. Furthermore, the real supply of assets must equal the real demand. This implies that $\int_0^1 g_i di = g^A$, $\int_0^1 b_i di = b^A$, $\int_0^1 a_i di = \bar{a}$, where $\bar{a}$ is the fixed supply of the risky asset.

A stationary equilibrium requires that g^A is constant, or $d(\phi G^A) = 0$.[8] This implies that

$$d(\phi G^A) = \phi dG^A + G^A d\phi = 0 \implies \frac{d\phi}{\phi} = -\frac{dG^A}{G^A} \implies \alpha_\phi = -\mu; \sigma_\phi = -\sigma$$

In other words, the price of the commodity in terms of the centralized market good is given as

$$\frac{d\phi}{\phi} = -\mu dt - \sigma dz \tag{34}$$

This implies that the expected rate of change in the purchasing power of the commodity is the negative expected rate of change in the supply of the commodity.[9] In addition, unexpected increases in the supply of the commodity have a negative impact on the rate of change in the commodity's purchasing power.

It follows that the equilibrium conditions from the maximization problem are given as

$$\rho + \mu = \theta \frac{1}{2}\left(\frac{u'(q)}{c'(q)} - 1\right) \tag{35}$$

$$\rho - \alpha_a = \theta \frac{1 - \chi_a}{2}\left(\frac{u'(q)}{c'(q)} - 1\right) \tag{36}$$

$$\rho - r = \theta \frac{1 - \chi_b}{2}\left(\frac{u'(q)}{c'(q)} - 1\right) \tag{37}$$

Equations (35)–(37) are sufficient to solve for q, α_a, and r. Finally, given that $\alpha_\phi = -\mu$ and $\sigma_\phi = -\sigma$, equation (33) is sufficient to solve for σ_a.

[8] In the commodity money literature, the price level depends on the supply and demand for the commodity. If the demand for the commodity is constant in equilibrium, as it is in this paper, then the real supply of the commodity is also constant over time. See the appendix.

[9] Note that Proposition 1 required making an assumption about an endogenous parameter, α_ϕ. This is equivalent to an assumption that $\mu > \frac{\theta}{2(1-\chi_a)} - \frac{\theta}{2} - \rho$.

Discussion

In the model above, I derived the equilibrium for a commodity money economy in which the commodity circulates alongside a risky asset and a riskless bond. In this subsection, I outline what this equilibrium implies about relative rates of return and the volatility of the risky asset. In addition, I show that the only way that commodity money is associated with the efficient allocation is either by divine coincidence or by policymakers' direct control over the supply of the commodity. This characteristic implies that there is a potential role for bank notes in improving the equilibrium allocation. In fact, a banking system in which banks are willing to redeem bank notes for some real quantity of the commodity implies that individuals earn an expected rate of return equal to the risk-free real interest rate under this type of monetary regime. As a result, this type of monetary regime is capable of producing an efficient equilibrium if the real interest rate is equal to the rate of time preference.

Rates of return. Note from equations (35)–(37) that

$$\rho + \mu = \frac{\rho - r}{1 - \chi_b}$$

$$\rho + \mu = \frac{\rho - \alpha_a}{1 - \chi_a}$$

and therefore

$$r = \chi_b \rho - (1 - \chi_b)\mu \tag{38}$$

$$\alpha_a = \chi_a \rho - (1 - \chi_a)\mu \tag{39}$$

The real rate of return on bonds and the expected real rate of return on the asset are functions of the rate of time preference, the expected rate of growth in the commodity supply, and the transaction costs of using bonds in pairwise meetings. Specifically, the model implies that an expected increase in the supply of commodity money reduces the rate of return on the commodity. As a result, individuals will substitute away from the commodity and into the riskless bond and the risky asset claims.

Furthermore, the volatility of the risky asset can also be determined from equation (33) using the solutions for α_ϕ, σ_ϕ, α_a, and r as

$$\sigma_a = \left(\frac{\chi_a}{\chi_b} - 1\right)\sigma \tag{40}$$

Thus, the volatility of the risky asset is a function of the relative liquidity of risky asset claims and riskless bonds, as well as the volatility of the supply of the commodity.

Can commodity money be efficient? Recall that the marginal value of each asset is determined, respectively, by equations (20)–(22). Combining these equations with equation (24) yields

$$V_g = \frac{1}{2}[(\lambda_1 - 1)c'(q)]\frac{\partial q}{\partial g_i} + 1$$

$$V_a = \frac{1}{2}[(\lambda_1 - 1)c'(q)]\frac{\partial q}{\partial a_i} + 1$$

$$V_b = \frac{1}{2}[(\lambda_1 - 1)c'(q)]\frac{\partial q}{\partial b_i} + 1$$

Recall that when individuals are matched to trade, the potential surplus from trade is $u(q)$-$c(q)$. The efficient allocation is achieved when this surplus is maximized (i.e., when $u'(q) = c'(q)$). From equation (24), this occurs when $\lambda_1 = 1$ and therefore $V_g = V_a = V_b = 1$. It follows from equations (15)–(17) that efficiency requires that commodity money, risky claims, and bonds have the same expected return and that this expected return is equal to the rate of time preference ($\alpha_\phi = \alpha_a = r = \rho$). An important question is whether this equilibrium is feasible when individuals are holding commodity money.

From the equilibrium conditions, α_ϕ = -μ. Thus, for commodity money to be held in equilibrium and for the efficient allocation to be feasible, it must be true that α_ϕ = -$\mu = \rho$. However, recall that μ is the expected growth rate of the commodity supply. No policymaker in the model has control over this variable. As a result, the only way that the efficient outcome can be achieved is either through divine coincidence or by policymakers having direct control over the supply of the commodity. Nonetheless, the important implication is that there is no guarantee that commodity money can generate the efficient allocation.

The role of bank notes. The limitations of commodity money with respect to achieving an efficient allocation lead somewhat naturally into a discussion of bank notes. In other words, if trading the actual commodity itself does not produce the efficient allocation, then perhaps the introduction of bank notes can improve equilibrium allocations. For example, something like a bank could emerge to construct a portfolio of assets by issuing bank notes in exchange for these assets. Why would a bank be willing to undertake this task? Conceivably, he bank could issue bank notes to purchase assets consistent with the arbitrage portfolio in the "No-Arbitrage Conditions" section of this paper. By doing so, the bank could construct a risk-free portfolio, thereby earning the riskless rate of return. As more additional banks entered the market, the market would converge to a competitive equilibrium in which banks earned zero profits. In this competitive equilibrium, banks would pay the risk-free rate of return to those holding bank notes.

While this reasoning explains the bank's motivation, it does not explain why individuals would willingly hold these bank notes or how the payment of interest on bank notes would occur. The answer is that the bank can promise to redeem bank notes on demand in the future for the commodity at a fixed price. By doing so, the bank is giving anyone holding a bank note a perpetual American call option for the commodity. Arbitrage pricing of options implies that this option will earn an expected rate of return equal to the real interest rate. This will produce the efficient equilibrium outcome if the real rate of return on the riskless bond is equal to the rate of time preference.[10] To see this conclusion, note that the bank note is an option to purchase the commodity at a fixed price. This option has no expiration date and can be exercised at any time. Thus, the option embodied in bank notes is a perpetual American option. To understand the implications, consider the following example.

Suppose that the commodity in question is gold. If G^A is the aggregate quantity of gold in circulation and ϕ is the price of the gold in terms of the centralized market good, then ϕG^A is the number of goods that an ounce of gold can purchase, or the goods price of gold. In the model, a stationary equilibrium requires that $d\phi G^A = 0$.

In a commodity money regime with bank notes, the denomination of the bank notes is *defined* in terms of the commodity. For example, suppose that bank notes were denominated in dollars, where the word *dollar* is defined such that $1 is to equal 1/20 oz. of gold. By definition, the official price of gold is $20 per ounce. Any bank issuing bank notes is willing to sell an ounce of gold to the public for $20. Thus, the dollar value of the supply of gold is $P_g G^A$, where P_g is the price of gold in terms of dollars. Defining P as the dollar price of the centralized market good, it follows that the value of the supply of gold in terms of goods is $\frac{P_g}{P} G^A = \phi G^A$. Since I have a

[10] A similar argument was made by Thompson (1974), although using different reasoning.

model of the long run, I know that $d\phi G^{A} = 0$ in equilibrium. However, by stating this equilibrium condition, I am abstracting from the very process of arbitrage that generates this condition. And it is this *process* of arbitrage that generates the option value associated with bank notes.

While the official price of gold in this example is \$20, it does not necessarily follow that we can write $\phi = \$20/P$, except as an equilibrium condition. The reason is that under a commodity standard, the price of the commodity is often determined in world markets. As a result, the market price can deviate from the official price. These deviations are the source of the option value. To see this, suppose that initially the market price and official price are the same. An individual then sells an ounce of gold to the bank for \$20. If the market price of gold rises to \$22, then the individual can go to the bank and redeem the \$20 note for one ounce of gold and then sell the gold for \$22, thereby pocketing a 10 percent profit. If on the other hand, the price of gold falls to \$18, then the individual can simply continue to hold the bank note without taking the corresponding loss.[11] This choice (1) creates an option value for those holding bank notes and (2) ensures that $P_g = \$20$ holds in equilibrium. It follows that anyone holding bank notes in this economy has the option to buy gold at a strike price of \$20 and sell the gold for the market price, thereby earning a profit of $P_g - \$20$. It is perhaps by now well understood from the work of Black and Scholes (1973), Merton (1973), and Cox, Ross, and Rubinstein (1979) that the expected rate of return on an option contract is the risk-free rate of return. Since bank notes represent a call option on the underlying commodity, they have an expected rate of return equal to the risk-free rate. Similarly, people holding the commodity (in this example, gold) hold a put

[11] Note, however, that arbitrage does work in the opposite direction. If the price falls to \$18, individuals have an incentive to buy gold on the world market and sell the gold to the bank in exchange for \$20 notes.

option that allows them to sell the commodity at a fixed price. The expected rate of return on the commodity is therefore also the real risk-free rate.

The intuition behind this result can be understood as follows for both the individual and the bank. Suppose that the economy consists of two goods, apples and the commodity, and that there are banks that issue bank notes denominated in dollars. As in the example above, the dollar is defined as 1/20 of an ounce of the commodity, so the official price of the commodity is $20. Today, one ounce of the commodity purchases one apple, and $20 buys one apple as well. Suppose that prices fluctuate from one day to the next. Tomorrow, one ounce of the commodity buys two apples, and $12.50 buys one apple. An individual can buy 1.6 apples directly with $20. Therefore, an individual holding dollars from yesterday to today earned a rate of return in terms of apples of 60 percent. However, the individual does not have to purchase apples with dollars directly. The individual could redeem the $20 for an ounce of the commodity from a bank and then use the ounce of the commodity to purchase two apples. This yields a rate of return of 100 percent in terms of apples. As a practical matter, even if the commodity doesn't circulate as a medium of exchange, if $12.50 buys one apple and one-half of an ounce of the commodity buys one apple, then the market price of the commodity is $25. So the individual can buy an ounce of the commodity from the bank for $20, sell the commodity on the market for $25, and purchase two apples.

Now consider a scenario in which the dollar price of apples falls to $8 and the price of apples in terms of the commodity remains the same. In this scenario, converting the bank notes to the commodity and then to apples still yields two apples. However, someone holding $20 can now purchase 2.5 apples using bank notes directly. So the individual is better off using bank notes directly, since this yields a rate of return of 150 percent in terms of apples. At the same time, an individual holding one ounce of the commodity is going to sell that ounce for $20 and then use the $20 to purchase 2.5 apples. The holder of the commodity benefits from being able to purchase dollars at a fixed price in this example, for the same reason as the individual holding notes in the previous example.

These examples illustrate that since the market price of the commodity in terms of dollars deviates from its official price, there are potential arbitrage opportunities; the price of apples in terms of the commodity can differ from the price of apples in terms of dollars, but there is an official fixed price between the commodity and the dollar. These arbitrage opportunities result in two long-term effects. First, over the long term, the arbitrage opportunities will disappear. When the market price of the commodity is high, individuals have an incentive to buy the commodity at the official price and sell the commodity on the market. This behavior tends to lower the price of the commodity in the market. Thus, the market price tends to revert to the official price over time. In fact, the work of Officer (1986) shows that, under the gold standard, the market price of gold deviated from the official price and that the gold market was efficient in minimizing such deviations.

The second long-term implication follows from the first. If it is possible for the market price of the commodity to differ from the official price, then one can construct a portfolio of the commodity and dollars capable of earning a real rate of return from arbitrage. If such arbitrage opportunities provide a rate of return in all possible circumstances that is higher than the risk-free rate of interest, then individuals will have an incentive to borrow at the risk-free real rate to finance the portfolio of the commodity and bank notes. Since this portfolio earns a return in all states of the world that is higher than the risk-free rate, this portfolio is like a perpetual money pump in the sense that the arbitrageur is able to earn a guaranteed profit with an initial net position of zero dollars. However, this situation will encourage additional people to construct similar portfolios. Doing so reduces the rate of return from arbitrage and bids up the real risk-free rate. Eventually, the expected rate of return on the portfolio will converge to the real risk-free rate.

For banks, a similar logic applies. Suppose that a bank issues bank notes as liabilities to finance purchases of the commodity and the risk-free bond. Since bank notes represent an option to purchase the commodity at a fixed price, the returns on the commodity and the bank notes are perfectly correlated. As a result, it is possible for the bank to perfectly hedge any price risk associated with the commodity. Competition in a free banking system will then drive the rate of return on the net worth of the bank, as well as the commodity and bank notes, to the risk-free rate of return.[12]

In each of these examples, the rate of return on each asset converges to the real risk-free rate because bank notes are a contingent claim on the commodity. In particular, bank notes are equivalent to an option to purchase the commodity at a fixed price. The contingent claim has one source of risk and one underlying asset. A key result in financial economics is that when these conditions are met, the market is both complete and arbitrage free, and each asset has an expected return equal to the real risk-free rate of interest.[13] Ross (2005, 12) explains the intuition:

> Merton pointed out that since the returns on the call were perfectly correlated with those on the underlying stock, a portfolio of the risk-free asset and the stock could be constructed at each point in time that would have identical returns to those on the call. To prevent arbitrage, then, a one-dollar investment in this replicating portfolio would be equivalent to a one-dollar investment in the call. . . .
>
> Cox and Ross observed that once it was known that arbitrage alone would determine the value of the call option, analysts were handing off the problem to the mathematicians prematurely. Since arbitrage determines the value, they argued that the value would be

[12] This is just an application of the pioneering approach to financial options presented by Merton (1973). For details, see the appendix.
[13] For a discussion, see Björk (2009, 121–22).

> determined by what they called risk-neutral valuation . . . The assumed expected return on the stock, μ [Author's note: α_{ϕ} in my example in the appendix], is irrelevant for valuation since in a risk-neutral world all assets have the riskless rate as their expected return.

Since bank notes represent a call option on the commodity, the intuition implies that bank notes and the underlying asset would similarly have an expected rate of return equal to the risk-free rate.

In contrast, when bank notes are not redeemable for some real quantity of a commodity, this mechanism no longer applies. The crucial characteristic of redeemable money is that it allows one unit of currency to be redeemable for a fixed quantity of the commodity. Fluctuations in the market price of the commodity relative to the official price generate an arbitrage opportunity. Thus, the option to redeem bank notes for the commodity has value. In contrast, an individual holding a $20 bill under an inconvertible paper money regime can redeem this bill for a $10 bill and two $5 bills or any asset with the same nominal value. As a result, inconvertible paper money does not have any sort of option value.

It is important to contrast this argument with a conventional understanding of fiat money. For example, there has long been a rate-of-return-dominance puzzle about money. In any equilibrium in which cash coexists with other assets that pay a higher rate of return, the marginal cost of holding each asset is equal to the marginal benefit. Since the pecuniary rates of return differ, there must be some nonpecuniary liquidity premium on cash that offsets the difference in pecuniary yields. One might assume that my argument about option value is just some variation of this result. However, that is not the case. When I say that bank notes and the commodity have an expected rate of return equal to the real risk-free rate, I do not mean that the bank is literally and explicitly paying a rate of return. I also do not mean to imply that this statement denotes some type of liquidity premium. My argument is that the pecuniary expected rate of return comes from arbitrage and is equal to the real risk-free rate because when there is one contingent claim with one underlying asset and one shock, the market is complete and arbitrage-free. A convertible bank note regime is therefore capable of implementing the Friedman rule automatically, whereas an economy with fiat currency is not.

Lessons for Monetary Policy

As I noted in the introduction, one argument for implementing a nominal income targeting regime for monetary policy is that it would replicate an important characteristic of free banking; namely, it would eliminate fluctuations in the economy owing to deviations between the money supply and money demand. I showed in the previous section that commodity money regimes with bank note issuance can potentially implement the Friedman rule automatically. As a result, an additional property of free banking has been previously ignored by advocates of nominal income targeting. In fact, the case for creating a monetary regime that replicates a free banking regime is therefore stronger than previously claimed. Whether or not monetary policy can replicate the experience of free banking is an open question. In this section, I use the model of commodity money from the previous section to draw lessons for implementing nominal income targeting to replicate a free banking system within the current institutional settings and within a market-based central banking regime.

Nominal Income Targeting within the Current Institutional Constraints

Conventional proposals to shift monetary policy toward nominal income targeting are largely agnostic about the target itself.[14] Regardless of whether they advocate targeting the level of nominal income or the growth rate, most proposals do not indicate a specific desired growth rate

[14] A notable exception is McCallum (1987, 1988), who advocates a rule of 3 percent growth. Sumner (2013) suggests a target nominal income growth rate of 3–5 percent.

for nominal income. However, the model above, as well as the broad literature on the Friedman rule, suggests that proposals to target nominal income need to take into account the rate of return on money balances. Commodity-based regimes in which banks issue convertible bank notes provide an expected return equal to the risk-free rate. This argument pertains to all commodity money regimes. If the risk-free rate is equal to the rate of time preference, then commodity-based regimes achieve the efficient steady-state equilibrium allocation. This suggests that monetary regimes in which bank notes are convertible into some commodity can achieve the efficient allocation without deflation.

Nonetheless, not all commodity-based monetary systems are created equal. Under a pure commodity money regime, the efficient allocation is only achievable by divine coincidence or by giving policymakers direct control over the supply of the commodity. The addition of convertible bank notes creates the conditions necessary to achieve the efficient equilibrium allocation. A commodity-based regime in which bank notes are redeemable for the commodity can come in two forms. The first is a regime in which the central bank has a monopoly to issue convertible bank notes. The second regime is one in which banks are free to enter the market and each issue their own convertible notes. While either of these regimes effectively creates a positive implicit expected rate of return on bank notes, the way in which the supply of bank notes adjusts to changes in the demand for bank notes is not identical.

In a free banking regime, banks are driven by their desire to maximize profit to expand the size of their balance sheet and therefore their note issuance. However, overissuing banks will observe a "reflux" of bank notes, thereby signaling that they need to reduce the supply of notes. Banks are driven by market forces to produce the precise quantity of bank notes that the public desires to hold, and the supply is therefore demand determined. Since the quantity of bank notes in circulation adjusts to changes in demand, a free banking system remains in "monetary equilibrium," thereby mitigating potential fluctuations in the economy (Hendrickson 2015).

In a regime in which a central bank issues convertible bank notes, the profit mechanism is absent. An overissue might lead to a "reflux" of notes to the central bank and therefore a drain on reserves, but a central bank has discretion over the reserve ratio. As long as the central bank does not fear an inability to honor its promise of convertibility, it might simply allow the reserve ratio to fluctuate. In fact, central banks might view the reserve ratio as an instrument of monetary policy. The shift from free banking to a central bank can potentially result in monetary disequilibria.

In the current institutional environment, central banks operate in a world of inconvertible paper money. The lack of convertibility means that bank notes no longer earn an implicit expected return equal to the risk-free rate. The lack of competitive note issuance means that there is a potential for monetary disequilibria if the central bank does not adjust the money supply in accordance with money demand. Advocates of nominal income targeting have emphasized the fact that stabilizing nominal income automatically stabilizes MV, according to the equation of exchange, and therefore eliminates monetary disequilibria in the same way that a free banking system does. However, advocates of nominal income targeting have not addressed the fact that the current institutional environment does not have interest-bearing bank notes, implicitly or explicitly. In such an environment, a central bank would need to adopt the Friedman rule, in which the central bank drives the nominal interest rate to zero through a deflationary policy. By setting the inflation rate equal to the negative rate of time preference, the central bank can achieve the efficient equilibrium allocation.

The strength of nominal income targeting is that it can achieve both of these characteristics of a free banking regime by setting a nominal income target that takes into account the desirability of the Friedman rule. In fact, an underappreciated element of Friedman's (1969) argument in favor of what has come to be known as the Friedman rule is that he reconciles this deflationary policy with his k-percent rule for the money supply. In particular, he argues that one could determine the desired constant rate of money supply growth by taking an estimate of the long-run trend in real income and subtracting the rate of deflation that would be necessary to implement the Friedman rule. A nominal income target would modify this argument by saying that one should simply use the same method to specify a target for nominal income growth. By doing so, the rule would not only implement the Friedman rule, but would also allow the money supply to adjust to offset changes in money demand.

It would be hard to isolate the degree to which implicitly interest-bearing bank notes contribute to economic welfare relative to the contribution of a supply of bank notes that is entirely demand determined, since the desirability of these outcomes draws on different definitions of welfare (Hendrickson 2015). Nonetheless, since both of these outcomes are welfare improving, and since nominal income targeting can potentially achieve them both, advocates of nominal income targeting would be wise to advocate particular targets for nominal income that take into account the desirability of the Friedman rule.

Nominal Income Targeting in a Market-Based Regime

Advocates of nominal income targeting have also proposed implementing such a target in the context of much different, market-based monetary regimes. For example, Sumner (1989, 2013) proposes targeting nominal income futures. In this proposal, the central bank would offer to buy and sell nominal income futures contracts at a price equal to one plus the target growth rate in nominal income. The contract would pay out a price equal to one plus the actual rate of growth in nominal income over the specified period of time. Whether an individual investor earned a profit or a loss on these contracts would depend on whether nominal income growth ended up above or below the target. Individuals who expect nominal income growth to be higher than its target could buy futures from the central bank. If their expectation were correct, they would earn a profit. Individuals who thought nominal income growth would come in below target could sell futures to the central bank at the official price. If these individuals proved correct, they would earn a profit. The buying and selling of these futures contracts would signal to the central bank whether the market expected nominal income to be above, below, or equal to target. Sumner suggests that the central bank could then use these signals to conduct parallel open market operations. For example, if the volume of futures contract purchases exceeds the volume of sales, this tells the central bank that the market thinks that the central bank is too expansionary. As a result, the central bank could conduct open market sales until this signal dissipated. Correspondingly, if the volume of sales of futures contracts exceeded the volume of purchases, then the central bank would conduct open market purchases to boost nominal income.[15]

To some extent, Sumner's proposal resembles a commodity money standard with convertibility in the sense that the central bank would stand ready to buy and sell an asset (a futures contract) on demand, at a fixed price. Nonetheless, there are important differences between nominal income futures targeting and a commodity standard with convertible bank notes.

To assess the proposal to target nominal income through a futures market, it is perhaps worthwhile to contrast this proposal with another market-based proposal with different, yet important, characteristics. In particular, consider the proposal of Thompson (1982) and Glasner

[15] There are issues related to the actual implementation of this sort of policy. I will not discuss those here, but the interested reader is referred to the discussion in Sumner (2013).

(1989).[16] The Thompson-Glasner plan proposes to stabilize an index of nominal wages through the use of indirect convertibility. To understand this concept, consider that under the gold standard with a central bank, the bank promises to redeem currency for a fixed quantity of gold. This stabilizes the price of gold such that, in equilibrium, the market price and official price of gold are the same. In theory, one could do the same thing by defining the dollar in terms of a quantity of labor. For example, suppose that all labor is homogeneous and the government defines the dollar as six minutes of labor. Thus, the official price of an hour of labor is $10. Of course, labor isn't traded continuously. In addition, it is not clear what it would mean to say that government is willing to buy and sell labor on demand. Nonetheless, this sort of system could employ indirect convertibility by having the government agree to buy and sell gold in the open market at the market price. To see how this would work in reality, consider the following example.

In reality, labor is not homogeneous. Thus, I could modify the definition of the dollar such that one dollar is *on average* equal to six minutes of labor. This implies that the average wage is $10 per hour. The government could then target an index of nominal wages to ensure that the average wage was equal to $10. Now, suppose that the current price of gold is $1000. Let's assume that any individual who wants an account can get an account with the government to buy and sell gold. Suppose an individual buys gold for the current market price of $1000 during the current calendar month. In the subsequent month, the wage index is released for the month the gold was purchased. Suppose the average nominal wage is $11. The individual purchased gold that they thought was worth 100 hours of labor. However, the higher wage index implies that the gold would actually buy less labor than the individual anticipated. As a result,

[16] See also Hendrickson (2018).

the government would credit the account of anyone who had purchased gold by $100 per ounce purchased. Anyone who had sold gold would have their account debited by $100 per ounce sold.

The important feature of this system is that it results in automatic stabilization of the nominal wage. If the nominal wage is expected to be higher than its target, then individuals will have an incentive to purchase gold. However, by purchasing gold, these individuals are pulling dollars out of the system, thereby reducing aggregate demand. This causes output, prices, and wages to decline and makes it more likely that the nominal wage index is equal to its target. Conversely, if individuals expected the nominal wage to be lower than the target, they would all have an incentive to sell gold to the government. Doing so would increase the supply of money in the system, raising output, prices, and wages—and, again, making it more likely that the nominal wage index would hit its target.

Both Sumner's proposal and the Thompson-Glasner proposal effectively outsource open market operations to the market. Under Sumner's plan, the central bank offers to buy and sell nominal income futures contracts. Under the Thompson-Glasner plan, the central bank promises to trade some type of asset, such as gold, and rebate or charge those who traded the asset with the government when the index deviates from target. Sumner's plan, by targeting nominal income, ensures that the money supply adjusts to changes in money demand. The Thompson-Glasner proposal ensures that the labor market clears. In a world in which monetary shocks are the sole source of fluctuations, these proposals are equivalent.

Despite these similarities, there is one crucial difference between the plans. The Thompson-Glasner plan, through indirect convertibility, makes the dollar convertible (indirectly) into something real, a particular quantity of labor. Under Sumner's plan, the dollar is not convertible into anything real. Unlike a commodity standard, however, a dollar is not equivalent to an option contract in either proposal. The reason is that individuals do not have the ability to redeem currency on demand for a quantity of a commodity at a fixed price. By buying and selling gold, individuals are essentially buying futures contracts. Since (1) currency is a claim to gold *of equivalent value* rather than a *particular quantity* of gold, and (2) only those who have bought or sold gold from the government are able to obtain a possible rate of return from changes in the wage index, currency lacks the option value of bank notes under a commodity regime.

Nonetheless, this difference has important implications for the efficiency of the long-run equilibrium. In the absence of a positive rate of return on currency, an efficient outcome requires implementing the Friedman rule. Under Sumner's proposal, this would require that the nominal income target have a deflationary bias. In contrast, the Thompson-Glasner proposal would effectively fix the average nominal wage. In the long run, the price level would have to adjust to clear the market for labor. Fluctuations in the labor market owing to nominal factors would be automatically adjusted through the government's outsourcing of open market operations. Fluctuations in the labor market caused by real factors would require that the price level adjust to clear the labor market. As a result, in a growing economy, the Thompson-Glasner plan would produce deflation naturally.[17]

Conclusion

Nominal income targeting has received increased attention from both economists and policymakers in recent years. One argument in favor of nominal income targeting is that it would replicate an important property of free banking regimes because it would automatically eliminate fluctuations in the economy due to monetary shocks. By eliminating monetary disequilibria,

[17] This result is the productivity norm discussed by Selgin (1997).

nominal income targeting contains an important stabilization property. However, little consideration has been given to the long-run efficiency properties of a nominal income target. In this paper, I have shown that pure commodity regimes, like their fiat money counterparts, are unlikely to produce the efficient long-run allocation. Free banking systems, by contrast, are potentially able to eliminate long-run inefficiencies because they offer an expected return on bank notes equal to the risk-free rate. Interest-bearing money makes it possible to automatically achieve an efficient outcome. This property bolsters the case for a nominal income target that closely replicates the characteristics of a free banking system. As I discussed, regardless of whether policy is conducted within the current institutional arrangements or through a market-based policy mechanism, a nominal income targeting regime must have a deflationary bias to completely replicate the desirable properties of a free banking regime.

Appendix

A Discussion on the Supply of the Commodity

In this paper, I assume that the supply of the commodity follows a geometric Brownian motion. Since the real value of the commodity in terms of the centralized market good is constant in equilibrium, the expected rate of return on the commodity is the negative expected rate of growth in the stock of the commodity. Some have criticized this assumption on the grounds that Hotelling's rule suggests that the rate of return on an exhaustible resource is equal to the real interest rate. If so, shouldn't the rate of return on the commodity be equal to the real interest rate? And if so, shouldn't any commodity regime produce the efficient outcome without appealing to option values? In this appendix I answer these questions in the following way.

First, I present a version of the Hotelling problem. I show that the maximization problem yields a condition in which the expected rate of return from leaving the resource in the ground is equal to the expected rate of return from extracting the resource, selling it, and investing the proceeds in the riskless asset. However, this does not imply that the rate of return on the asset itself is equal to real interest rate. Rather, it implies that the rate of return on the *option* to extract the resource is equal to the real interest rate. Whether or not the rate of return on the resource itself is equal to the real interest rate depends on supply and demand. Furthermore, I show that when the price process is stochastic, this complicates the model further *despite* starting with the assumption that the rate of return on the option is equal to the real interest rate.

Second, I point out that the implication that the relative price of the commodity is determined by the supply and demand for the commodity is consistent across models of commodity money. I first discuss White's (1999) stock-flow equilibrium model and then Barro's (1979) model to demonstrate the consistency of the assumption of my model with the literature.

Hotelling's rule. Hotelling's rule can be understood as follows. Suppose that there is a firm that owns an exhaustible resource that needs to be mined. Suppose that the price of an exhaustible resource is given as

$$p(t) = p_0 e^{\mu t}$$

where p_0 is the initial price of the resource and μ is the growth rate of the price over time.

Suppose that total cost is linear in the quantity of production. It follows that the profit per unit of output can be written as

$$\text{Profit}(t) = p(t) - c$$

where c is the average (and marginal) cost of output.

The owner of this exhaustible resource would like to choose the optimal point in time to extract the resource. If $\mu > 0$, then the price of the asset is rising over time. Since I have assumed that the evolution of the price over time is certain, the owner could choose the price at which to extract the resource or the point in time. Assume that we are initially at time $t = 0$. Without specifying the pricing process, the problem facing the owner is as follows:

$$\max_T e^{-rT} [p(T) - c]$$

In other words, the value of extracting the resource in time period T is $p(T)$-c. However, since this date is (possibly) in the future, we need to consider the present value of the extraction. There is an inherent tradeoff here. If $\mu > 0$, then the price is rising over time. The longer the owner waits to extract the resource, the greater the profit per unit, but the longer the owner has to wait for that benefit. The owner therefore wants to optimally balance this tradeoff. Maximizing this expression with respect to T yields

$$r[p(T) - c] = p'(T)$$

This condition implies that the rate of return from leaving the resource in the ground (the appreciation in the price) is equal to the rate of return from extracting the resource, selling it, and investing the proceeds in the riskless asset.

With the given price process, the maximization problem can be rewritten as

$$\max_T e^{(\mu - r)T} p_0 - e^{-rT} c$$

The first-order condition for maximization is that

$$(r - \mu) e^{\mu T} p_o = rc \tag{41}$$

The second-order condition is

$$\mu (r - \mu) e^{\mu T} p_o < 0$$

Note that this only holds if $\mu < r$. The reason is simple. If the price of the resource is rising faster than the rate of interest, then it is optimal to leave the resource in the ground forever.

Assuming that the second-order condition is satisfied, the first-order condition can be rewritten as

$$T^* = \max \left\{ \frac{1}{\mu} \ln \left(\frac{rc}{(r - \mu) p_0} \right), 0 \right\} \tag{42}$$

Note that no part of this problem pins down the rate of return on the resource itself. In fact, the resource is assumed to be growing at an exogenous rate, μ.

Furthermore, consider what happens as μ approaches r. From equation (42), in the limit as μ goes to r, the extraction time for the resource becomes infinite. The resource is never extracted.

So what rate of return is being determined here? It is the rate of return on the *option* to extract the resource. To see why, let's denote the value of this option as $V(P)$. If the rate of return on this option is equal to the real interest rate, it follows that

$$\frac{1}{dt}\frac{dV}{V} = r$$

Now, from the price process we know that

$$\frac{dP}{P} = \mu dt$$

So,

$$r = \frac{1}{dt}\frac{V'(P)dP}{V(P)} = \frac{1}{dt}\frac{V'(P)\mu P dt}{V(P)}$$

Or,

$$rV(P) = V'(P)\mu P$$

Now, there is some threshold for the price at which the resource will be extracted. Denote this price as P^*. It follows that at the threshold for extraction, the value of the option must be equal to the value of extraction. This implies that $V(P^*) = P^*$-C. Plugging this into the equation above yields

$$r(P^* - C) = \mu P^*$$

This implies that the rate of return from leaving the resource in the ground, μP^*, is equal to the rate of return from extracting the resource, selling it, and buying the risk-free asset, $r(P^*\text{-}c)$. This condition is equivalent to saying that the rate of return on the option to extract the asset is equal to the real interest rate.

Consider also an example in which the price of the commodity is given as

$$p(t) = p_0 e^{\mu t + \sigma z}$$

where σ is the standard deviation and z is a Wiener process. This dramatically changes the implications because it is now no longer possible to choose the point in time at which to extract the resource. Instead, the objective is to choose a threshold for the price. Once the price hits the threshold, then the resource is extracted.

Let $p(t) = p_0 e^{\mu t + \sigma z}$ and $V(p)$ denote the value of the option to extract the resource. Now let's assume that the rate of return on the option is equal to r as was the case under certainty.

$$\frac{1}{dt} E \frac{dV}{V(p)} = r$$

where E is the expectations operator.

Expanding this equation yields

$$\frac{1}{dt} E \left[V'(p)dp + \frac{1}{2} V''(p)(dp)^2 \right] - rV(p) = 0$$

Using the path specified for the price, it follows that

$$dp = \mu p_0 e^{\mu t + \sigma Z} dt + \sigma p_0 e^{\mu t + \sigma z} dz$$

Dividing both sides by p yields

$$\frac{dp}{p} = \mu dt + \sigma dz$$

where $dz = \epsilon\sqrt{dt}$ and ϵ is drawn from a standard normal process. Plugging this into the value equation yields

$$\frac{1}{2}\sigma^2 p^2 V''(p) + \mu p V'(p) - rV(p) = 0$$

If we impose the boundary conditions on $V(p)$ that the option is worthless when the price goes to 0 and that the benefit when option is exercised is p^*-c, there is a known solution to this differential equation that can be written as

$$V(p) = \left(\frac{p}{p^*}\right)^\beta (p^* - c)$$

Maximizing this expression with respect to p^* yields a threshold for p at which the resource is extracted that will be a markup above the average cost of extraction. Furthermore, the time period in which the resource is extracted is $T = \inf\{t \geq 0 | p \geq p^*\}$. Again, nothing in this problem pins down the expected rate of return on the resource as it did in the certainty case. As a result, there is no guarantee that the expected rate of return on the commodity in the paper is determined by Hotelling's rule.

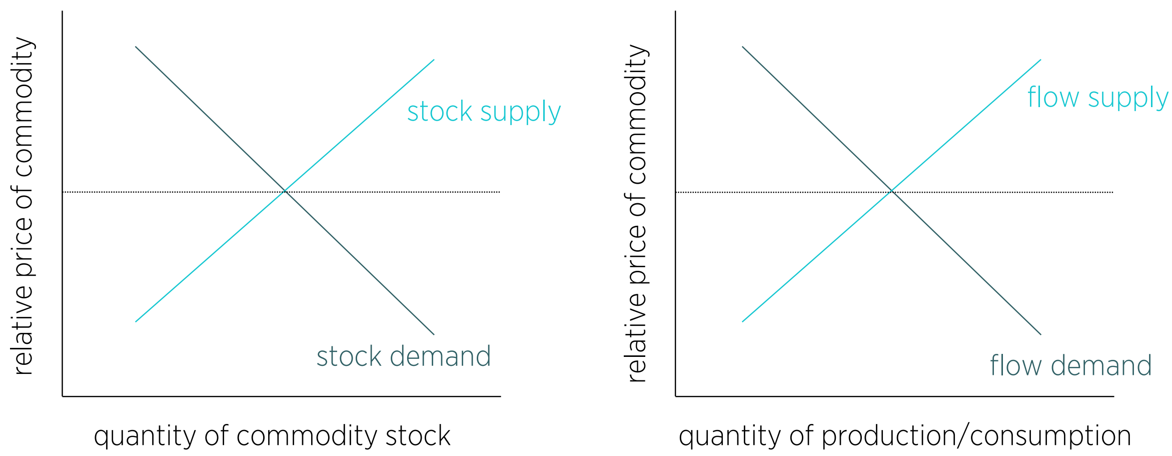
Note that in a Hotelling problem, the rate of return on the resource is taken *as given* by producers, as is the real interest rate. This is a competitive market assumption, and in competitive markets, the actual rate of return is determined by supply and demand. In Hotelling's original model, he simply assumes a demand curve for the resource and then solves the production problem. In this paper, the demand for the resource depends on its role as a medium of exchange. When the exhaustible resource is also a medium of exchange, the expected rate of return on the resource will depend not only on the production decisions of the owner of the exhaustible resource (the supply decision), but also on the portfolio decisions of individuals who decide how much of the resource to hold over time when other, imperfect substitutes are available (the demand decision). In my model, the demand for the resource is constant in real terms in the long-run steady-state equilibrium. If the supply of the resource is increasing, then the price of the resource must be correspondingly declining. If the supply of the resource is decreasing, then the price of the resource must be correspondingly increasing. This pins down the rate of return on the resource. This is consistent with an assumption of competitive markets.

Finally, I demonstrate that my result regarding the determination of relative price of the commodity is consistent with the previous literature below by outlining White's stock-flow model of a commodity standard and Barro's model of the gold standard.

White's model of a stock-flow equilibrium. White's (1999) model is a stock-flow model. It can be represented graphically. Panel A of figure 1 shows the stock supply and stock demand for the commodity. The relative price of the commodity is on the vertical axis, and the stock of the commodity is on the horizontal axis. Panel B shows the flow supply and flow demand for the commodity. The relative supply of the commodity is on the vertical axis, and the flow quantity is on the horizontal axis. In equilibrium, the relative price of the commodity clears both markets.

Figure 1. White's Stock-Flow Model

Panel A. Stock Equilibrium ***Panel B. Flow Equilibrium***



Fluctuations in the stock supply and the stock demand cause temporary deviations of the relative price of the commodity. To see why, consider an increase in the stock demand for the commodity. This shifts the stock demand curve to the right. The relative price of the commodity increases. This increase in the relative price of the commodity leads to a deviation between the flow supply and flow demand. In particular, the higher relative price of the commodity leads to an increase in the flow supply in excess of the flow demand, which increases the stock supply of the commodity. As the stock supply of the commodity increases, the relative price of the commodity returns to its original equilibrium point.

In contrast, shifts in the flow supply or flow demand of the commodity cause a permanent change in the relative price of the commodity. To understand why, suppose that there is an increase in the flow supply of the commodity. This leads to an increase in the stock supply of the commodity. As a result, the relative price of the commodity declines. Unlike in the case of fluctuations in the stock supply or stock demand, there is no corrective mechanism that moves the relative price back to its original equilibrium. Thus, changes in the flow supply or flow demand result in permanent changes in the relative price of the commodity.

Since my model focuses on the long run, I am only concerned with flows, and I assume that these flows are exogenous. Specifically, I assume that the only source of changes in the long run are changes in the flow supply of gold. Take the assumption that the stock of the commodity follows this geometric Brownian motion:

$$\frac{dG^A}{G^A} = \mu dt + \sigma dz$$

The stock of gold changes over time owing to changes in the flow supply. What the assumption of geometric Brownian motion implies is that, on average, the flow supply of the commodity increases the stock by μ percent. However, there are unexpected deviations from this average (e.g., mines produce less than anticipated, or new mines are discovered). Notice that in the context of White's model, increases and decreases in the flow supply of commodity are a primary source of long-run fluctuations in the purchasing power, or relative price, of the commodity. This is also true of my model. In the steady state, the real demand for the commodity is constant. As a result, the real supply must also be constant. Since the quantity of the commodity is changing over time, the relative price of commodity must change accordingly.

Barro's model and neutrality. The assumption that real commodity balances are neutral is consistent with the assumption of constant commodity demand and fluctuating commodity supply. For example, consider the model of Barro (1979). He assumes that money demand is a function of income, such that $\frac{M^d}{P} = kY$, where M is the quantity of bank notes, P is the price level, k is the fraction of income held in bank notes, and Y is income. Furthermore, the supply of bank notes is assumed to be $M^s = (1/r)P_g G$, where $r \in (0,1)$ is the reserve ratio, P_g is the price of gold, and G is the supply of gold. Note that since there are no bank notes in the model, $r = 1$ because the money supply is equal to the commodity supply. Equilibrium requires that money supply equals money demand. Thus,

$$PkY = P_g G$$

Solving for the real supply of gold yields

$$\frac{P_g}{P} G = \phi G = kY$$

If k and Y are constant over time (as they are in my model), then the real supply of gold is constant over time as well.

On Option Values

To understand why both bank notes and the underlying commodity earn the risk-free rate of return, consider a simplified, consolidated bank balance sheet in which the bank holds the commodity and bonds as assets and issues bank notes as liabilities. Let $\mathcal{N}$ denote the net worth of the bank, s_1 denote the quantity of the commodity reserves held by the bank, and s_2 denote the quantity of bonds held by the bank. Let C denote the value of bank notes. It follows that the net worth of the bank can be written as

$$\mathcal{N} = s_1\phi + s_2 P_b - C(\phi)$$

Assuming that the quantities of the commodity and the bond are constant, differentiating this expression yields

$$d\mathcal{N} = s_1 d\phi + s_2 dP_b - dC$$

The value of the bank notes is a function of the purchasing power of the underlying commodity. Thus, using Ito's Lemma, it follows that

$$dC = C'(\phi)d\phi + \frac{1}{2}C''(\phi)(d\phi)^2$$

Using the conjectured process for ϕ from the model, and ignoring terms of higher order than dt, it follows that

$$dC = \left(\alpha_\phi \phi C'(\phi) + \frac{1}{2}C''(\phi)\phi^2\sigma_\phi^2\right)dt + \sigma_\phi \phi C'(\phi)dz$$

To simplify things, let's rewrite this expression as

$$dC = \alpha_c C dt + \sigma_c C dz$$

It follows that the evolution of the bank's net worth can be written as

$$d\mathcal{N} = (s_1\alpha_\phi \phi + s_2 r P_b - \alpha_c C)dt + (s_1\sigma_\phi \phi - \sigma_c C)dz$$

Note that since the underlying commodity is subject to random shocks to its purchasing power, this represents a risk to the bank. However, by issuing bank notes, the bank is selling call options on the underlying commodity. Since money demand is determined by the portfolio choices of the note holders, the bank cannot choose how many bank notes to issue. The supply of bank notes is demand determined. Nonetheless, the bank can choose the reserve ratio ($\phi s_1/C$) that eliminates the risk to the bank. In particular, note that the risk to net worth can be eliminated if

$$s_1 \sigma_\phi \phi - \sigma_c C = 0$$

This occurs if the bank sets the reserve ratio such that

$$\frac{\phi s_1}{C} = \frac{\sigma_c}{\sigma_\phi}$$

If the bank sets this as the reserve ratio, the net worth of the bank is riskless. If this is true and the rate of return on the net worth of the bank is greater than the risk-free rate, then the bank is a money pump. In other words, the bank can earn arbitrage profits without bearing any risk. With free banking, there are no barriers to entry. As a result, one would expect that competitors would emerge to try to capture some of this risk-free arbitrage. Ultimately, however, as banks enter the market, they compete away the risk-free arbitrage opportunities. It follows that in equilibrium, the rate of return on the net worth of the bank must equal the risk-free rate of return:

$$\frac{dN}{N} = r dt$$

Using the "evolution of net worth" equation and the reserve ratio that eliminates risk, it follows that

$$dN = \left(\frac{\sigma_c}{\sigma_\phi} C \alpha_\phi + s_2 r P_b - \alpha_c C\right) dt = rN dt = \left(r \frac{\sigma_c}{\sigma_\phi} C + s_2 r P_b - rC\right) dt$$

Or,

$$\frac{\alpha_\phi - r}{\sigma_\phi} = \frac{\alpha_c - r}{\sigma_c}$$

Recall that

$$\alpha_c = \frac{1}{C}\left(\alpha_\phi \phi C'(\phi) + \frac{1}{2}C''(\phi)\phi^2\sigma_\phi^2\right)$$

$$\sigma_c = \frac{1}{C}\sigma_\phi \phi C'(\phi)$$

Plugging this into the no-arbitrage condition yields

$$\frac{\alpha_\phi - r}{\sigma_\phi} = \frac{\frac{1}{C}\left(\alpha_\phi \phi C'(\phi) + \frac{1}{2}C''(\phi)\phi^2\sigma_\phi^2\right) - r}{\frac{1}{C}\sigma_\phi \phi C'(\phi)}$$

Simplifying this expression yields

$$(\alpha_\phi - r)\phi C'(\phi) = \alpha_\phi \phi C'(\phi) + \frac{1}{2}C''(\phi)\phi^2\sigma^2 - rC$$

Or,

$$\frac{1}{2}\sigma_\phi^2\phi^2 C''(\phi) + r\phi C'(\phi) - rC = 0$$

Note that this is the Black-Scholes equation with $\partial C/\partial t = 0$ since the option does not expire. The solution to this second-order differential equation determines the value of the bank notes. Note that the solution to the equation does not depend at all on the expected rate of return on the underlying asset, α_ϕ. In fact, the option is valued as though both the option and the underlying commodity have an expected rate of return equal to the risk-free rate.

To illustrate this, suppose that the underlying commodity has an expected return equal to the risk-free rate such that

$$d\phi = r\phi dt + \sigma\phi dz$$

If the option also earns a risk-free expected return, then

$$E(dC/C) = rdt$$

Recall that

$$dC = C'(\phi)d\phi + \frac{1}{2}C''(\phi)(d\phi)^2$$

So,

$$rC = \frac{1}{dt}E(dC) = \frac{1}{dt}E\left(C'(\phi)d\phi + \frac{1}{2}C''(\phi)(d\phi)^2\right) = \frac{1}{2}\sigma_\phi^2\phi^2C''(\phi) + r\phi C'(\phi)$$

Or, rearranging,

$$\frac{1}{2}\sigma_\phi^2\phi^2C''(\phi) + r\phi C'(\phi) - rC = 0$$

This is the identical equation derived above.

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